Trigonometric Substitution

In finding the area of a circle or an ellipse, an integral of the form $\int \sqrt{a^2 - x^2}\, dx$ arises, where $a > 0$. If it were $\int x\sqrt{a^2 - x^2}\, dx$, the substitution $u = a^2 - x^2$ would be effective but, as it stands, $\int \sqrt{a^2 - x^2}\, dx$ is more difficult. If we change the variable from x to θ by the substitution $x = a \sin \theta$, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta} = a|\cos \theta|$$

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) and the substitution $x = a \sin \theta$ (the old variable is a function of the new one).

In general we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse. To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one. In this case, if we replace u by x and x by t in the Substitution Rule (Equation 5.5.4), we obtain

$$\int f(x)\, dx = \int f(g(t))g'(t)\, dt$$

This kind of substitution is called *inverse substitution.*

We can make the inverse substitution $x = a \sin \theta$ provided that it defines a one-to-one function. This can be accomplished by restricting θ to lie in the interval $[-\pi/2, \pi/2]$.

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities. In each case the restriction on θ is imposed to ensure that the function that defines the substitution is one-to-one. (These are the same intervals used in Appendix D in defining the inverse functions.)

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

EXAMPLE 1 Evaluate $\displaystyle\int \frac{\sqrt{9 - x^2}}{x^2}\, dx$.

SOLUTION Let $x = 3 \sin \theta$, where $-\pi/2 \leqslant \theta \leqslant \pi/2$. Then $dx = 3 \cos \theta\, d\theta$ and

$$\sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2\theta} = \sqrt{9 \cos^2\theta} = 3|\cos \theta| = 3 \cos \theta$$

(Note that $\cos \theta \geqslant 0$ because $-\pi/2 \leqslant \theta \leqslant \pi/2$.) Thus, the Inverse Substitution Rule gives

$$\begin{aligned}
\int \frac{\sqrt{9 - x^2}}{x^2}\, dx &= \int \frac{3 \cos \theta}{9 \sin^2\theta}\, 3 \cos \theta\, d\theta \\
&= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \cot^2\theta\, d\theta \\
&= \int (\csc^2\theta - 1)\, d\theta \\
&= -\cot \theta - \theta + C
\end{aligned}$$

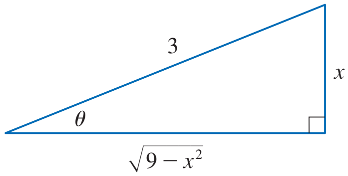


FIGURE 1

$\sin\theta = \dfrac{x}{3}$

Since this is an indefinite integral, we must return to the original variable x. This can be done either by using trigonometric identities to express $\cot\theta$ in terms of $\sin\theta = x/3$ or by drawing a diagram, as in Figure 1, where θ is interpreted as an angle of a right triangle. Since $\sin\theta = x/3$, we label the opposite side and the hypotenuse as having lengths x and 3. Then the Pythagorean Theorem gives the length of the adjacent side as $\sqrt{9-x^2}$, so we can simply read the value of $\cot\theta$ from the figure:

$$\cot\theta = \frac{\sqrt{9-x^2}}{x}$$

(Although $\theta > 0$ in the diagram, this expression for $\cot\theta$ is valid even when $\theta < 0$.) Since $\sin\theta = x/3$, we have $\theta = \sin^{-1}(x/3)$ and so

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

EXAMPLE 2 Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

SOLUTION Solving the equation of the ellipse for y, we get

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \qquad \text{or} \qquad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

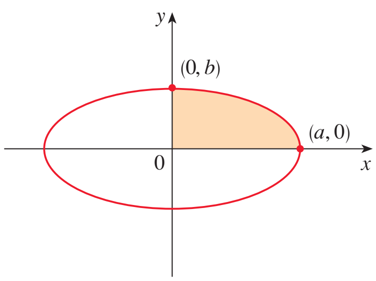


FIGURE 2

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Because the ellipse is symmetric with respect to both axes, the total area A is four times the area in the first quadrant (see Figure 2). The part of the ellipse in the first quadrant is given by the function

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad 0 \leqslant x \leqslant a$$

and so

$$\tfrac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

To evaluate this integral we substitute $x = a\sin\theta$. Then $dx = a\cos\theta\,d\theta$. To change the limits of integration we note that when $x = 0$, $\sin\theta = 0$, so $\theta = 0$; when $x = a$, $\sin\theta = 1$, so $\theta = \pi/2$. Also

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a|\cos\theta| = a\cos\theta$$

since $0 \leqslant \theta \leqslant \pi/2$. Therefore

$$\begin{aligned} A &= 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = 4\frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\,d\theta \\ &= 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\tfrac{1}{2}(1 + \cos 2\theta)\,d\theta \\ &= 2ab\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left(\frac{\pi}{2} + 0 - 0\right) \\ &= \pi ab \end{aligned}$$

We have shown that the area of an ellipse with semiaxes a and b is πab. In particular, taking $a = b = r$, we have proved the famous formula that the area of a circle with radius r is πr^2.

NOTE ◦ Since the integral in Example 2 was a definite integral, we changed the limits of integration and did not have to convert back to the original variable x.

EXAMPLE 3 Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

SOLUTION Let $x = 2\tan\theta$, $-\pi/2 < \theta < \pi/2$. Then $dx = 2\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+4} = \sqrt{4(\tan^2\theta+1)} = \sqrt{4\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$$

Thus, we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta\cdot 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

To evaluate this trigonometric integral we put everything in terms of $\sin\theta$ and $\cos\theta$:

$$\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

Therefore, making the substitution $u = \sin\theta$, we have

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2+4}} &= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{du}{u^2}\\ &= \frac{1}{4}\left(-\frac{1}{u}\right) + C = -\frac{1}{4\sin\theta} + C\\ &= -\frac{\csc\theta}{4} + C\end{aligned}$$

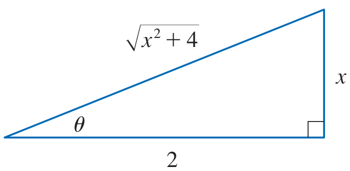


FIGURE 3

$\tan\theta = \frac{x}{2}$

We use Figure 3 to determine that $\csc\theta = \sqrt{x^2+4}/x$ and so

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x} + C$$

EXAMPLE 4 Find $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,dx$.

SOLUTION It would be possible to use the trigonometric substitution $x = 2\tan\theta$ here (as in Example 3). But the direct substitution $u = x^2 + 4$ is simpler, because then $du = 2x\,dx$ and

$$\int \frac{x}{\sqrt{x^2+4}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+4} + C$$

NOTE ▫ Example 4 illustrates the fact that even when trigonometric substitutions are possible, they may not give the easiest solution. You should look for a simpler method first.

EXAMPLE 5 Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$, where $a > 0$.

SOLUTION We let $x = a\sec\theta$, where $0 < \theta < \pi/2$ or $\pi < \theta < 3\pi/2$. Then $dx = a\sec\theta\tan\theta\,d\theta$ and

$$\sqrt{x^2-a^2} = \sqrt{a^2(\sec^2\theta-1)} = \sqrt{a^2\tan^2\theta} = a|\tan\theta| = a\tan\theta$$

Therefore

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-a^2}} &= \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\,d\theta\\ &= \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C\end{aligned}$$

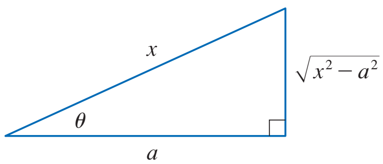


FIGURE 4
$\sec\theta = \dfrac{x}{a}$

The triangle in Figure 4 gives $\tan\theta = \sqrt{x^2 - a^2}/a$, so we have

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C$$
$$= \ln\left|x + \sqrt{x^2 - a^2}\right| - \ln a + C$$

Writing $C_1 = C - \ln a$, we have

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + C_1$$

EXAMPLE 6 Find $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\,dx$.

SOLUTION First we note that $(4x^2 + 9)^{3/2} = \left(\sqrt{4x^2 + 9}\right)^3$ so trigonometric substitution is appropriate. Although $\sqrt{4x^2 + 9}$ is not quite one of the expressions in the table of trigonometric substitutions, it becomes one of them if we make the preliminary substitution $u = 2x$. When we combine this with the tangent substitution, we have $x = \frac{3}{2}\tan\theta$, which gives $dx = \frac{3}{2}\sec^2\theta\,d\theta$ and

$$\sqrt{4x^2 + 9} = \sqrt{9\tan^2\theta + 9} = 3\sec\theta$$

When $x = 0$, $\tan\theta = 0$, so $\theta = 0$; when $x = 3\sqrt{3}/2$, $\tan\theta = \sqrt{3}$, so $\theta = \pi/3$.

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\,dx = \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\,\tfrac{3}{2}\sec^2\theta\,d\theta$$
$$= \tfrac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta = \tfrac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$
$$= \tfrac{3}{16}\int_0^{\pi/3} \frac{1 - \cos^2\theta}{\cos^2\theta}\,\sin\theta\,d\theta$$

Now we substitute $u = \cos\theta$ so that $du = -\sin\theta\,d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \pi/3$, $u = \frac{1}{2}$.
Therefore

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\,dx = -\tfrac{3}{16}\int_1^{1/2} \frac{1 - u^2}{u^2}\,du = \tfrac{3}{16}\int_1^{1/2} (1 - u^{-2})\,du$$
$$= \tfrac{3}{16}\left[u + \frac{1}{u}\right]_1^{1/2} = \tfrac{3}{16}\left[\left(\tfrac{1}{2} + 2\right) - (1 + 1)\right] = \tfrac{3}{32}$$

EXAMPLE 7 Evaluate $\displaystyle\int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx$.

SOLUTION We can transform the integrand into a function for which trigonometric substitution is appropriate by first completing the square under the root sign:

$$3 - 2x - x^2 = 3 - (x^2 + 2x) = 3 + 1 - (x^2 + 2x + 1)$$
$$= 4 - (x + 1)^2$$

This suggests that we make the substitution $u = x + 1$. Then $du = dx$ and $x = u - 1$, so

$$\int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx = \int \frac{u - 1}{\sqrt{4 - u^2}}\,du$$

We now substitute $u = 2\sin\theta$, giving $du = 2\cos\theta\,d\theta$ and $\sqrt{4-u^2} = 2\cos\theta$, so

$$\begin{aligned}\int \frac{x}{\sqrt{3-2x-x^2}}\,dx &= \int \frac{2\sin\theta - 1}{2\cos\theta}\,2\cos\theta\,d\theta \\ &= \int (2\sin\theta - 1)\,d\theta \\ &= -2\cos\theta - \theta + C \\ &= -\sqrt{4-u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C \\ &= -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C\end{aligned}$$

Exercises

A Click here for answers. S Click here for solutions.

1–3 ■ Evaluate the integral using the indicated trigonometric substitution. Sketch and label the associated right triangle.

1. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\,dx;\quad x = 3\sec\theta$

2. $\displaystyle\int x^3\sqrt{9-x^2}\,dx;\quad x = 3\sin\theta$

3. $\displaystyle\int \frac{x^3}{\sqrt{x^2+9}}\,dx;\quad x = 3\tan\theta$

4–30 ■ Evaluate the integral.

4. $\displaystyle\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$

5. $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt$

6. $\displaystyle\int_0^2 x^3\sqrt{x^2+4}\,dx$

7. $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$

8. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx$

9. $\displaystyle\int \frac{dx}{\sqrt{x^2+16}}$

10. $\displaystyle\int \frac{t^5}{\sqrt{t^2+2}}\,dt$

11. $\displaystyle\int \sqrt{1-4x^2}\,dx$

12. $\displaystyle\int_0^1 x\sqrt{x^2+4}\,dx$

13. $\displaystyle\int \frac{\sqrt{x^2-9}}{x^3}\,dx$

14. $\displaystyle\int \frac{du}{u\sqrt{5-u^2}}$

15. $\displaystyle\int \frac{x^2}{(a^2-x^2)^{3/2}}\,dx$

16. $\displaystyle\int \frac{dx}{x^2\sqrt{16x^2-9}}$

17. $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx$

18. $\displaystyle\int \frac{dx}{[(ax)^2-b^2]^{3/2}}$

19. $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\,dx$

20. $\displaystyle\int \frac{t}{\sqrt{25-t^2}}\,dt$

21. $\displaystyle\int_0^{2/3} x^3\sqrt{4-9x^2}\,dx$

22. $\displaystyle\int_0^1 \sqrt{x^2+1}\,dx$

23. $\displaystyle\int \sqrt{5+4x-x^2}\,dx$

24. $\displaystyle\int \frac{dt}{\sqrt{t^2-6t+13}}$

25. $\displaystyle\int \frac{1}{\sqrt{9x^2+6x-8}}\,dx$

26. $\displaystyle\int \frac{x^2}{\sqrt{4x-x^2}}\,dx$

27. $\displaystyle\int \frac{dx}{(x^2+2x+2)^2}$

28. $\displaystyle\int \frac{dx}{(5-4x-x^2)^{5/2}}$

29. $\displaystyle\int x\sqrt{1-x^4}\,dx$

30. $\displaystyle\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt$

31. (a) Use trigonometric substitution to show that

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right) + C$$

(b) Use the hyperbolic substitution $x = a\sinh t$ to show that

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

These formulas are connected by Formula 3.9.3.

32. Evaluate

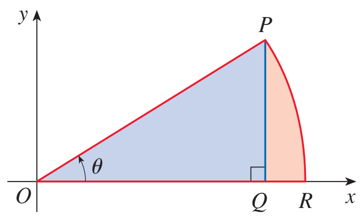
$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\,dx$$

(a) by trigonometric substitution.
(b) by the hyperbolic substitution $x = a\sinh t$.

33. Find the average value of $f(x) = \sqrt{x^2 - 1}/x$, $1 \leqslant x \leqslant 7$.

34. Find the area of the region bounded by the hyperbola $9x^2 - 4y^2 = 36$ and the line $x = 3$.

35. Prove the formula $A = \frac{1}{2}r^2\theta$ for the area of a sector of a circle with radius r and central angle θ. [*Hint:* Assume $0 < \theta < \pi/2$ and place the center of the circle at the origin so it has the equation $x^2 + y^2 = r^2$. Then A is the sum of the area of the triangle POQ and the area of the region PQR in the figure.]



36. Evaluate the integral

$$\int \frac{dx}{x^4\sqrt{x^2 - 2}}$$

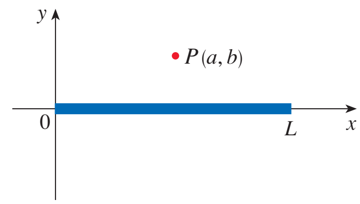
Graph the integrand and its indefinite integral on the same screen and check that your answer is reasonable.

37. Use a graph to approximate the roots of the equation $x^2\sqrt{4 - x^2} = 2 - x$. Then approximate the area bounded by the curve $y = x^2\sqrt{4 - x^2}$ and the line $y = 2 - x$.

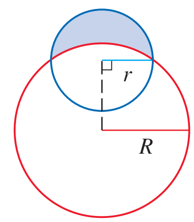
38. A charged rod of length L produces an electric field at point $P(a, b)$ given by

$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\varepsilon_0(x^2 + b^2)^{3/2}}\, dx$$

where λ is the charge density per unit length on the rod and ε_0 is the free space permittivity (see the figure). Evaluate the integral to determine an expression for the electric field $E(P)$.



39. Find the area of the crescent-shaped region (called a *lune*) bounded by arcs of circles with radii r and R. (See the figure.)



40. A water storage tank has the shape of a cylinder with diameter 10 ft. It is mounted so that the circular cross-sections are vertical. If the depth of the water is 7 ft, what percentage of the total capacity is being used?

41. A torus is generated by rotating the circle $x^2 + (y - R)^2 = r^2$ about the x-axis. Find the volume enclosed by the torus.

Answers

S Click here for solutions.

1. $\sqrt{x^2-9}/(9x)+C$ **3.** $\frac{1}{3}(x^2-18)\sqrt{x^2+9}+C$

5. $\pi/24+\sqrt{3}/8-\frac{1}{4}$ **7.** $-\sqrt{25-x^2}/(25x)+C$

9. $\ln\left(\sqrt{x^2+16}+x\right)+C$ **11.** $\frac{1}{4}\sin^{-1}(2x)+\frac{1}{2}x\sqrt{1-4x^2}+C$

13. $\frac{1}{6}\sec^{-1}(x/3)-\sqrt{x^2-9}/(2x^2)+C$

15. $\left(x/\sqrt{a^2-x^2}\right)-\sin^{-1}(x/a)+C$ **17.** $\sqrt{x^2-7}+C$

19. $\ln\left|\left(\sqrt{1+x^2}-1\right)/x\right|+\sqrt{1+x^2}+C$ **21.** $\frac{64}{1215}$

23. $\frac{9}{2}\sin^{-1}((x-2)/3)+\frac{1}{2}(x-2)\sqrt{5+4x-x^2}+C$

25. $\frac{1}{3}\ln\left|3x+1+\sqrt{9x^2+6x-8}\right|+C$

27. $\frac{1}{2}[\tan^{-1}(x+1)+(x+1)/(x^2+2x+2)]+C$

29. $\frac{1}{4}\sin^{-1}x^2+\frac{1}{4}x^2\sqrt{1-x^4}+C$

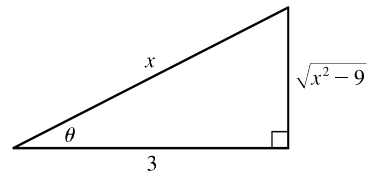
33. $\frac{1}{6}\left(\sqrt{48}-\sec^{-1}7\right)$ **37.** 0.81, 2; 2.10

39. $r\sqrt{R^2-r^2}+\pi r^2/2-R^2\arcsin(r/R)$ **41.** $2\pi^2Rr^2$

Solutions: Trigonometric Substitution

1. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 3\sec\theta\tan\theta\, d\theta$ and

$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9(\sec^2\theta - 1)} = \sqrt{9\tan^2\theta}$$
$$= 3\left|\tan\theta\right| = 3\tan\theta \text{ for the relevant values of } \theta.$$

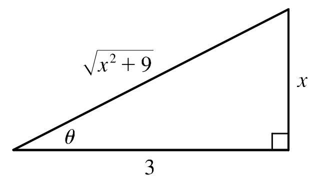


$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{1}{9\sec^2\theta \cdot 3\tan\theta} 3\sec\theta\tan\theta\,d\theta = \tfrac{1}{9}\int \cos\theta\,d\theta = \tfrac{1}{9}\sin\theta + C = \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

Note that $-\sec(\theta+\pi) = \sec\theta$, so the figure is sufficient for the case $\pi \le \theta < \frac{3\pi}{2}$.

3. Let $x = 3\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 3\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = \sqrt{9(\tan^2\theta + 1)} = \sqrt{9\sec^2\theta}$$
$$= 3\left|\sec\theta\right| = 3\sec\theta \text{ for the relevant values of } \theta.$$



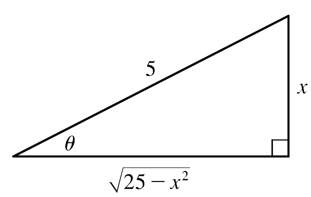
$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{3^3\tan^3\theta}{3\sec\theta} 3\sec^2\theta\,d\theta = 3^3\int \tan^3\theta\sec\theta d\theta = 3^3\int \tan^2\theta\tan\theta\sec\theta\,d\theta$$
$$= 3^3\int\left(\sec^2\theta - 1\right)\tan\theta\sec\theta\,d\theta = 3^3\int\left(u^2-1\right)du \qquad [u = \sec\theta,\ du = \sec\theta\tan\theta\,d\theta]$$
$$= 3^3\left(\tfrac{1}{3}u^3 - u\right) + C = 3^3\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right) + C = 3^3\left[\frac{1}{3}\frac{\left(x^2+9\right)^{3/2}}{3^3} - \frac{\sqrt{x^2+9}}{3}\right] + C$$
$$= \tfrac{1}{3}\left(x^2+9\right)^{3/2} - 9\sqrt{x^2+9} + C \quad \text{or} \quad \tfrac{1}{3}\left(x^2-18\right)\sqrt{x^2+9} + C$$

5. Let $t = \sec\theta$, so $dt = \sec\theta\tan\theta\,d\theta$, $t = \sqrt{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$, and $t = 2 \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta\tan\theta}\sec\theta\tan\theta\,d\theta = \int_{\pi/4}^{\pi/3}\frac{1}{\sec^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3}\cos^2\theta\,d\theta$$
$$= \textstyle\int_{\pi/4}^{\pi/3}\frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$
$$= \tfrac{1}{2}\left[\left(\tfrac{\pi}{3} + \tfrac{1}{2}\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{\pi}{4} + \tfrac{1}{2}\cdot 1\right)\right] = \tfrac{1}{2}\left(\tfrac{\pi}{12} + \tfrac{\sqrt{3}}{4} - \tfrac{1}{2}\right) = \tfrac{\pi}{24} + \tfrac{\sqrt{3}}{8} - \tfrac{1}{4}$$

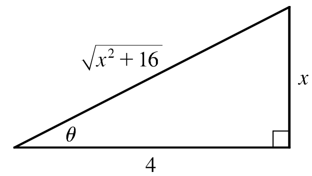
7. Let $x = 5\sin\theta$, so $dx = 5\cos\theta\,d\theta$. Then

$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \int \frac{1}{5^2\sin^2\theta\cdot 5\cos\theta}5\cos\theta\,d\theta$$
$$= \tfrac{1}{25}\int\csc^2\theta\,d\theta = -\tfrac{1}{25}\cot\theta + C$$
$$= -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C$$



9. Let $x = 4\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 4\sec^2\theta\,d\theta$ and

$$\begin{aligned}\sqrt{x^2+16} &= \sqrt{16\tan^2\theta + 16} = \sqrt{16(\tan^2\theta+1)}\\ &= \sqrt{16\sec^2\theta} = 4\,|\sec\theta|\\ &= 4\sec\theta \text{ for the relevant values of } \theta.\end{aligned}$$

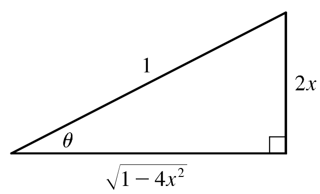


$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+16}} &= \int \frac{4\sec^2\theta\,d\theta}{4\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C_1\\ &= \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C_1 = \ln\left|\sqrt{x^2+16}+x\right| - \ln|4| + C_1\\ &= \ln\left(\sqrt{x^2+16}+x\right) + C, \text{ where } C = C_1 - \ln 4.\end{aligned}$$

(Since $\sqrt{x^2+16}+x > 0$, we don't need the absolute value.)

11. Let $2x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $x = \frac{1}{2}\sin\theta$,

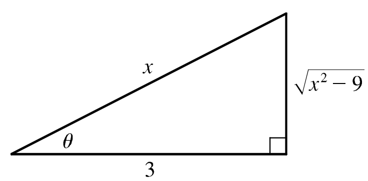
$dx = \frac{1}{2}\cos\theta\,d\theta$, and $\sqrt{1-4x^2} = \sqrt{1-(2x)^2} = \cos\theta$.



$$\begin{aligned}\int \sqrt{1-4x^2}\,dx &= \int \cos\theta\left(\tfrac{1}{2}\cos\theta\right)d\theta = \tfrac{1}{4}\int(1+\cos 2\theta)\,d\theta\\ &= \tfrac{1}{4}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{4}(\theta + \sin\theta\cos\theta) + C\\ &= \tfrac{1}{4}\left[\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\right] + C\end{aligned}$$

13. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

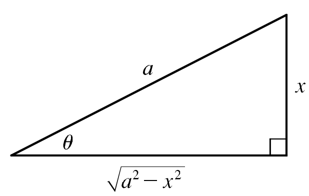
$dx = 3\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$, so



$$\begin{aligned}\int \frac{\sqrt{x^2-9}}{x^3}\,dx &= \int \frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\tan\theta\,d\theta = \frac{1}{3}\int\frac{\tan^2\theta}{\sec^2\theta}\,d\theta\\ &= \tfrac{1}{3}\int\sin^2\theta\,d\theta = \tfrac{1}{3}\int\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{6}\theta - \tfrac{1}{12}\sin 2\theta + C = \tfrac{1}{6}\theta - \tfrac{1}{6}\sin\theta\cos\theta + C\\ &= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\,\frac{\sqrt{x^2-9}}{x}\,\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C\end{aligned}$$

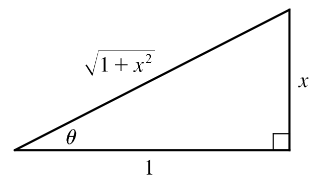
15. Let $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = a\cos\theta\,d\theta$ and

$$\begin{aligned}\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} &= \int \frac{a^2\sin^2\theta\,a\cos\theta\,d\theta}{a^3\cos^3\theta} = \int \tan^2\theta\,d\theta\\ &= \int\left(\sec^2\theta - 1\right)d\theta = \tan\theta - \theta + C\\ &= \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C\end{aligned}$$



17. Let $u = x^2 - 7$, so $du = 2x\,dx$. Then $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx = \frac{1}{2}\int\frac{1}{\sqrt{u}}\,du = \tfrac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2-7} + C$.

19. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\, d\theta$ and $\sqrt{1+x^2} = \sec\theta$, so



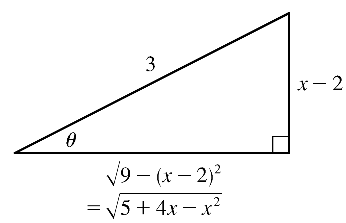
$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\,d\theta$$
$$= \int (\csc\theta + \sec\theta\tan\theta)\,d\theta$$
$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C \quad \text{[by Exercise 39 in } \textit{Additional Topics: Trigonometric Integrals}\text{]}$$
$$= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2} + C$$

21. Let $u = 4 - 9x^2 \quad\Rightarrow\quad du = -18x\,dx$. Then $x^2 = \frac{1}{9}(4-u)$ and

$$\int_0^{2/3} x^3\sqrt{4-9x^2}\,dx = \int_4^0 \tfrac{1}{9}(4-u)u^{1/2}\left(-\tfrac{1}{18}\right)du = \tfrac{1}{162}\int_0^4\left(4u^{1/2} - u^{3/2}\right)du$$
$$= \tfrac{1}{162}\left[\tfrac{8}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\right]_0^4 = \tfrac{1}{162}\left[\tfrac{64}{3} - \tfrac{64}{5}\right] = \tfrac{64}{1215}$$

Or: Let $3x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

23. $5 + 4x - x^2 = -(x^2 - 4x + 4) + 9 = -(x-2)^2 + 9$. Let $x - 2 = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so $dx = 3\cos\theta\,d\theta$. Then



$$\int\sqrt{5+4x-x^2}\,dx = \int\sqrt{9-(x-2)^2}\,dx = \int\sqrt{9-9\sin^2\theta}\,3\cos\theta\,d\theta$$
$$= \int\sqrt{9\cos^2\theta}\,3\cos\theta\,d\theta = \int 9\cos^2\theta\,d\theta$$
$$= \tfrac{9}{2}\int(1+\cos 2\theta)\,d\theta = \tfrac{9}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C$$
$$= \tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta + C = \tfrac{9}{2}\theta + \tfrac{9}{4}(2\sin\theta\cos\theta) + C$$
$$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{5+4x-x^2}}{3} + C$$
$$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C$$

25. $9x^2 + 6x - 8 = (3x+1)^2 - 9$, so let $u = 3x + 1$, $du = 3dx$. Then $\displaystyle\int\frac{dx}{\sqrt{9x^2+6x-8}} = \int\frac{\frac{1}{3}\,du}{\sqrt{u^2-9}}$. Now let $u = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $du = 3\sec\theta\tan\theta\,d\theta$ and $\sqrt{u^2-9} = 3\tan\theta$, so

$$\int\frac{\frac{1}{3}\,du}{\sqrt{u^2-9}} = \int\frac{\sec\theta\tan\theta\,d\theta}{3\tan\theta} = \tfrac{1}{3}\int\sec\theta\,d\theta = \tfrac{1}{3}\ln|\sec\theta + \tan\theta| + C_1 = \tfrac{1}{3}\ln\left|\frac{u+\sqrt{u^2-9}}{3}\right| + C_1$$
$$= \tfrac{1}{3}\ln\left|u + \sqrt{u^2-9}\right| + C = \tfrac{1}{3}\ln\left|3x + 1 + \sqrt{9x^2+6x-8}\right| + C$$

27. $x^2 + 2x + 2 = (x+1)^2 + 1$. Let $u = x + 1$, $du = dx$. Then

$$\int\frac{dx}{(x^2+2x+2)^2} = \int\frac{du}{(u^2+1)^2} = \int\frac{\sec^2\theta\,d\theta}{\sec^4\theta} \quad \left[\begin{array}{c}\text{where } u = \tan\theta,\ du = \sec^2\theta\,d\theta, \\ \text{and } u^2+1 = \sec^2\theta\end{array}\right]$$
$$= \int\cos^2\theta\,d\theta = \tfrac{1}{2}\int(1+\cos 2\theta)\,d\theta = \tfrac{1}{2}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{1}{2}\left[\tan^{-1}u + \frac{u}{1+u^2}\right] + C = \frac{1}{2}\left[\tan^{-1}(x+1) + \frac{x+1}{x^2+2x+2}\right] + C$$

29. Let $u = x^2$, $du = 2x\,dx$. Then

$$\int x\sqrt{1-x^4}\,dx = \int \sqrt{1-u^2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int \cos\theta\cdot\cos\theta\,d\theta \quad \left[\begin{array}{c}\text{where } u=\sin\theta,\ du=\cos\theta\,d\theta, \\ \text{and } \sqrt{1-u^2}=\cos\theta\end{array}\right]$$

$$= \tfrac{1}{2}\int \tfrac{1}{2}(1+\cos 2\theta)d\theta = \tfrac{1}{4}\theta + \tfrac{1}{8}\sin 2\theta + C = \tfrac{1}{4}\theta + \tfrac{1}{4}\sin\theta\cos\theta + C$$

$$= \tfrac{1}{4}\sin^{-1}u + \tfrac{1}{4}u\sqrt{1-u^2} + C = \tfrac{1}{4}\sin^{-1}(x^2) + \tfrac{1}{4}x^2\sqrt{1-x^4} + C$$

31. (a) Let $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $\sqrt{x^2+a^2} = a\sec\theta$ and

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta\,d\theta}{a\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C_1 = \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1$$

$$= \ln\left(x+\sqrt{x^2+a^2}\right) + C \quad \text{where } C = C_1 - \ln|a|$$

(b) Let $x = a\sinh t$, so that $dx = a\cosh t\,dt$ and $\sqrt{x^2+a^2} = a\cosh t$. Then

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\cosh t\,dt}{a\cosh t} = t + C = \sinh^{-1}\frac{x}{a} + C.$$

33. The average value of $f(x) = \sqrt{x^2-1}/x$ on the interval $[1, 7]$ is

$$\frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{6}\int_0^\alpha \frac{\tan\theta}{\sec\theta}\cdot\sec\theta\tan\theta\,d\theta \quad \left[\begin{array}{c}\text{where } x=\sec\theta,\ dx=\sec\theta\tan\theta\,d\theta, \\ \sqrt{x^2-1}=\tan\theta, \text{ and } \alpha=\sec^{-1}7\end{array}\right]$$

$$= \tfrac{1}{6}\int_0^\alpha \tan^2\theta\,d\theta = \tfrac{1}{6}\int_0^\alpha (\sec^2\theta - 1)\,d\theta$$

$$= \tfrac{1}{6}\Big[\tan\theta - \theta\Big]_0^\alpha = \tfrac{1}{6}(\tan\alpha - \alpha)$$

$$= \tfrac{1}{6}\left(\sqrt{48} - \sec^{-1}7\right)$$

35. Area of $\triangle POQ = \frac{1}{2}(r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2\sin\theta\cos\theta$. Area of region $PQR = \int_{r\cos\theta}^{r}\sqrt{r^2-x^2}\,dx$. Let $x = r\cos u \;\Rightarrow\; dx = -r\sin u\,du$ for $\theta \le u \le \frac{\pi}{2}$. Then we obtain

$$\int \sqrt{r^2-x^2}\,dx = \int r\sin u\,(-r\sin u)\,du = -r^2\int \sin^2 u\,du = -\tfrac{1}{2}r^2(u - \sin u\cos u) + C$$

$$= -\tfrac{1}{2}r^2\cos^{-1}(x/r) + \tfrac{1}{2}x\sqrt{r^2-x^2} + C$$

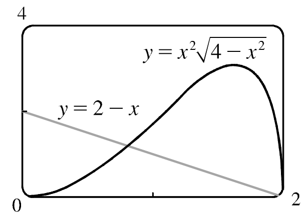
so

$$\text{area of region } PQR = \tfrac{1}{2}\left[-r^2\cos^{-1}(x/r) + x\sqrt{r^2-x^2}\right]_{r\cos\theta}^{r}$$

$$= \tfrac{1}{2}\left[0 - \left(-r^2\theta + r\cos\theta\, r\sin\theta\right)\right]$$

$$= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta\cos\theta$$

and thus, (area of sector POR) = (area of $\triangle POQ$) + (area of region PQR) $= \frac{1}{2}r^2\theta$.



37. From the graph, it appears that the curve $y = x^2\sqrt{4-x^2}$ and the line $y = 2 - x$ intersect at about $x = 0.81$ and $x = 2$, with $x^2\sqrt{4-x^2} > 2 - x$ on $(0.81, 2)$. So the area bounded by the curve and the line is $A \approx \int_{0.81}^{2}\left[x^2\sqrt{4-x^2} - (2-x)\right]dx = \int_{0.81}^{2} x^2\sqrt{4-x^2}\,dx - \left[2x - \frac{1}{2}x^2\right]_{0.81}^{2}$.
To evaluate the integral, we put $x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\,d\theta$, $x = 2 \;\Rightarrow\; \theta = \sin^{-1}1 = \frac{\pi}{2}$, and $x = 0.81 \;\Rightarrow\; \theta = \sin^{-1}0.405 \approx 0.417$. So

$$\int_{0.81}^{2} x^2\sqrt{4-x^2}\,dx \approx \int_{0.417}^{\pi/2} 4\sin^2\theta\,(2\cos\theta)(2\cos\theta\,d\theta) = 4\int_{0.417}^{\pi/2}\sin^2 2\theta\,d\theta = 4\int_{0.417}^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta$$
$$= 2\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_{0.417}^{\pi/2} = 2\left[\left(\tfrac{\pi}{2} - 0\right) - \left(0.417 - \tfrac{1}{4}(0.995)\right)\right] \approx 2.81$$

Thus, $A \approx 2.81 - \left[\left(2\cdot 2 - \frac{1}{2}\cdot 2^2\right) - \left(2\cdot 0.81 - \frac{1}{2}\cdot 0.81^2\right)\right] \approx 2.10$.

39. Let the equation of the large circle be $x^2 + y^2 = R^2$. Then the equation of the small circle is $x^2 + (y-b)^2 = r^2$, where $b = \sqrt{R^2 - r^2}$ is the distance between the centers of the circles. The desired area is

$$A = \int_{-r}^{r}\left[\left(b + \sqrt{r^2-x^2}\right) - \sqrt{R^2-x^2}\right]dx = 2\int_0^r\left(b + \sqrt{r^2-x^2} - \sqrt{R^2-x^2}\right)dx$$
$$= 2\int_0^r b\,dx + 2\int_0^r\sqrt{r^2-x^2}\,dx - 2\int_0^r\sqrt{R^2-x^2}\,dx$$

The first integral is just $2br = 2r\sqrt{R^2-r^2}$. To evaluate the other two integrals, note that

$$\int\sqrt{a^2-x^2}\,dx = \int a^2\cos^2\theta\,d\theta \quad [x = a\sin\theta,\ dx = a\cos\theta\,d\theta] \quad = \tfrac{1}{2}a^2\int(1+\cos 2\theta)\,d\theta$$
$$= \tfrac{1}{2}a^2\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{2}a^2(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a} + C = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C$$

so the desired area is

$$A = 2r\sqrt{R^2-r^2} + \left[r^2\arcsin(x/r) + x\sqrt{r^2-x^2}\right]_0^r - \left[R^2\arcsin(x/R) + x\sqrt{R^2-x^2}\right]_0^r$$
$$= 2r\sqrt{R^2-r^2} + r^2\left(\tfrac{\pi}{2}\right) - \left[R^2\arcsin(r/R) + r\sqrt{R^2-r^2}\right] = r\sqrt{R^2-r^2} + \tfrac{\pi}{2}r^2 - R^2\arcsin(r/R)$$

41. We use cylindrical shells and assume that $R > r$. $x^2 = r^2 - (y-R)^2 \;\Rightarrow\; x = \pm\sqrt{r^2-(y-R)^2}$, so $g(y) = 2\sqrt{r^2-(y-R)^2}$ and

$$V = \int_{R-r}^{R+r} 2\pi y\cdot 2\sqrt{r^2-(y-R)^2}\,dy = \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du \quad [\text{where } u = y - R]$$
$$= 4\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\,du + 4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du \quad \left[\begin{array}{c}\text{where } u = r\sin\theta,\ du = r\cos\theta\,d\theta\\ \text{in the second integral}\end{array}\right]$$
$$= 4\pi\left[-\tfrac{1}{3}\left(r^2-u^2\right)^{3/2}\right]_{-r}^{r} + 4\pi R\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\,d\theta = -\tfrac{4\pi}{3}(0-0) + 4\pi Rr^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$
$$= 2\pi Rr^2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta = 2\pi Rr^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2\pi^2 Rr^2$$

Another method: Use washers instead of shells, so $V = 8\pi R\int_0^r\sqrt{r^2-y^2}\,dy$ as in Exercise 6.2.39(a), but evaluate the integral using $y = r\sin\theta$.